In any triangle with usual notations, prove that

$$(a+2r)(b+2r)(c+2r) \geq 2R^3(\sqrt{3}+5).$$

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Since $2(\sqrt{3}+5) = \left(\sqrt{3}+1\right)^3$ then the inequality can be rewritten as

(1) $\quad (a+2r)(b+2r)(c+2r) \geq \left(R\left(\sqrt{3}+1\right)\right)^3.$

First note that for any triangle inequality of the problem is not right.

Counterexample for obtuse triangle:

Let $a = \sqrt{3}, b = c = 1.$ Then $R = 1,\ s = \dfrac{2+\sqrt{3}}{2}, r = \dfrac{2\sqrt{3}-3}{2},$

$(a+2r)(b+2r)(c+2r) = \left(1+2\cdot\dfrac{2\sqrt{3}-3}{2}\right)^2\left(\sqrt{3}+2\cdot\dfrac{2\sqrt{3}-3}{2}\right) =$

$\left(2\sqrt{3}-2\right)^2\left(3\sqrt{3}-3\right) = 12\left(\sqrt{3}-1\right)^3,\ R\left(\sqrt{3}+1\right) = \sqrt{3}+1$ and

$\sqrt[3]{12}\left(\sqrt{3}-1\right) < \sqrt{3}+1 \Leftrightarrow \sqrt[3]{12} < \dfrac{\sqrt{3}+1}{\sqrt{3}-1} \Leftrightarrow \sqrt[3]{12} < \dfrac{\left(\sqrt{3}+1\right)^2}{2} \Leftrightarrow$

$2\sqrt[3]{12} < \left(\sqrt{3}+1\right)^2 \Leftrightarrow \sqrt[3]{12} < \sqrt{3}+2 \Leftrightarrow 12 < \left(\sqrt{3}+2\right)^3 \Leftarrow 12 < 3^3.$

Counterexample for right triangle:

Let $a = 5, b = 3, c = 4.$ Then $R = \dfrac{5}{2},\ r = \dfrac{b+c-a}{2} = 1,$

$(a+2r)(b+2r)(c+2r) = (3+2)(4+2)(5+2) = 210,$

$2R^3\left(3\sqrt{3}+5\right) = 2\cdot\left(\dfrac{5}{2}\right)^3\left(3\sqrt{3}+5\right) = \dfrac{125\left(3\sqrt{3}+5\right)}{4} > \dfrac{125(3+5)}{4} = 250.$

Since a right triangle can be considered as limit state of an acute tringle then we can conclude that the inequality of the problem isn't holds in the set of any acute triangles. But since it is right if for equilateral tringle (in that case we have $R = 2r, a = b = c = 2r\sqrt{3}\ , (a+2r)^3 = \left(2r\sqrt{3}+2r\right)^3 = \left(\sqrt{3}+1\right)^3 8r^3 = R^3\left(\sqrt{3}+1\right)^3$)

I desided to check as **appropriate variant of original problem** the following problem:

In any triangle with usual notations, prove that

(2) $\quad (a+2r)(b+2r)(c+2r) \geq 16r^3\left(3\sqrt{3}+5\right) = \left(2r\left(\sqrt{3}+1\right)\right)^3.$

Proof of inequality (**2**).

By replacing in Huygens Inequality $(x+1)(y+1)(z+1) \geq \left((xyz)^{1/3}+1\right)^3$

(x,y,z) with $\left(\dfrac{a}{2r}, \dfrac{b}{2r}, \dfrac{b}{2r}\right)$ we obtain

$$\left(\frac{a}{2r}+1\right)\left(\frac{b}{2r}+1\right)\left(\frac{b}{2r}+1\right) \geq \left(\left(\frac{a}{2r}\cdot\frac{b}{2r}\cdot\frac{b}{2r}\right)^{1/3}+1\right)^3 \Leftrightarrow$$

$$(a+2r)(b+2r)(c+2r) \geq \left((abc)^{1/3}+2r\right)^3.$$

Since $(abc)^{1/3}+2r \geq 2r\left(\sqrt{3}+1\right) \Leftrightarrow (abc)^{1/3} \geq 2r\sqrt{3} \Leftrightarrow abc > 24r^3\sqrt{3} \Leftrightarrow$

$4Rrs > 24r^3\sqrt{3} \Leftrightarrow Rs > 6r^2\sqrt{3}$, where latter inequality holds because $R \geq 2r$ and $s \geq 3\sqrt{3}\,r$ then $(a+2r)(b+2r)(c+2r) \geq \left(2r\left(\sqrt{3}+1\right)\right)^3.$

Remark. **Limit counterxample**:

The fact that the inequality of the problem isn't holds for any triangle easily

follows from the consideration of a degenerate isosceles triangle (as the limiting case of an obtuse isosceles triangle):
If $a = b$ and $c = 2a$ then $R = \infty, r = 0$ and (**1**) isn$'$t holds.